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Diophantine Inequalities with Mixed Powers

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Let $\{\lambda_i\}_{i=1}^s$ $(s \geqslant 2)$ be a finite sequence of non-zero real numbers, not all of the same sign and in which not all the ratios λ_i/λ_j are rational. A given sequence of positive integers $\{n_i\}_{i=1}^s$ is said to have property (P) ((P^*) respectively) if for any $\{\lambda_i\}_{i=1}^s$ and any real number η, there exists a positive constant σ, depending on $\{\lambda_i\}_{i=1}^s$ and $\{n_i\}_{i=1}^s$ only, so that the inequality $|\eta + \sum_{i=1}^s \lambda_i x_i^{n_i}| < (\max x_i)^{-\sigma}$ has infinitely many solutions in positive integers (primes respectively) $x_1, x_2, \ldots, x_s$. In this paper, we prove the following result: Given a sequence of positive integers $\{n_i\}_{i=1}^\infty$, a necessary and sufficient condition that, for any positive integer j, there exists an integer s, depending on $\{n_i\}_{i=j}^\infty$ only, such that $\{n_i\}_{i=j}^{j+s-1}$ has property (P) (or (P^*)), is that $\sum_{i=1}^\infty n_i^{-1} = \infty$. These are parallel to some striking results of G. A. Freĭman, E. J. Scourfield and K. Thanigasalam.

1. INTRODUCTION

The following remarkable theorem was stated by Freĭman [4] in 1949, and proved by Scourfield [6, Theorem 1] in 1960:

Given a sequence $\{n_i\}_{i=1}^\infty$ of integers with $2 \leqslant n_1 \leqslant n_2 \leqslant \cdots$, a necessary and sufficient condition that, for any positive integer j, there exists a positive integer r such that all sufficiently large positive integers N are representable in the form

$$N = x_1^{n_j} + x_2^{n_{j+1}} + \cdots + x_r^{n_{j+r-1}},$$

where $x_1, x_2, \ldots, x_r$ are positive integers, is that $\sum_{i=1}^\infty n_i^{-1} = \infty$.

Later, in 1966, Thanigasalam [7, Theorem 2] was able to prove the corresponding result with all the integers x replaced by primes.

Throughout, $\{\lambda_i\}_{i=1}^s$ $(s \geqslant 2)$ is a finite sequence of non-zero real numbers,

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not all of the same sign and in which not all the ratios λ_i/λ_j are rational. A given sequence of positive integers $\{n_i\}_{i=1}^{s}$ is said to have *property* (P) $((P^*)$ *respectively*) if *for any* $\{\lambda_i\}_{i=1}^{s}$ *and any real number* η, *there exists a positive constant* σ, *depending on* $\{\lambda_i\}_{i=1}^{s}$ *and* $\{n_i\}_{i=1}^{s}$ *only, so that the inequality* $|\eta+\sum_{i=1}^{s}\lambda_i x_i^{n_i}|<(\max x_i)^{-\sigma}$ *has infinitely many solutions in positive integers* (*primes respectively*) $x_1, x_2, \ldots, x_s$.

The object of this paper is to consider diophantine inequalities with mixed powers and obtain results which are parallel to those of Freĭman, Scourfield, and Thanigasalam. More precisely, our main result is

THEOREM 1. *Given a sequence of positive integers* $\{n_i\}_{i=1}^{\infty}$, *a necessary and sufficient condition that, for any positive integer* j, *there exists an integer* s, *depending on* $\{n_i\}_{i=j}^{\infty}$ *only, such that* $\{n_i\}_{i=j}^{j+s-1}$ *has property* (P) (*or* (P^*)), *is that* $\sum_{i=1}^{\infty} n_i^{-1}=\infty$.

We shall first prove the following two theorems from which Theorem 1 follows easily.

THEOREM 2. *If* $n_1, n_2, \ldots, n_s$ *are positive integers such that* $\sum_{i=1}^{s} n_i^{-1}<1$, *then there exist real numbers* η *and* $\varepsilon>0$ *and a sequence* $\{\lambda_i\}_{i=1}^{s}$ *so that the inequality*

$$\left|\eta+\sum_{i=1}^{s}\lambda_i x_i^{n_i}\right|<\varepsilon \tag{1.1}$$

is not solvable in positive integers (*and hence primes*) $x_1, x_2, \ldots, x_s$.

For any positive integer k, if $s(k)$ is the least s for which the inequality $|\eta+\sum_{i=1}^{s}\lambda_i x_i^{k}|<\varepsilon$ is solvable for any real numbers η and $\varepsilon>0$, then our Theorem 2 shows that $s(k)\geqslant k$.

THEOREM 3. *If* $\{n_i\}_{i=1}^{s}$ *is a sequence of positive integers satisfying*

(i)

$$1\leqslant n_1\leqslant n_2\leqslant\cdots\leqslant n_s \tag{1.2}$$

and

(ii) *there are three integers* a, b, c *such that*

$$3\leqslant a,\qquad a+3\leqslant b,\qquad b+3\leqslant c,\qquad c+3\leqslant s,$$

$$\sum_{i=3}^{a}(4+n_i)^{-1}\geqslant\tfrac{1}{2},\qquad \sum_{i=a+3}^{b}(4+n_i)^{-1}\geqslant\tfrac{1}{2}, \tag{1.3}$$

$$\prod_{i=b+3}^{c}(1-n_i^{-1})<(1+4n_a^{-1})\,\theta_{n_a}$$

and

$$\prod_{i=c+3}^{s} (1 - n_i^{-1}) < (1 + 4n_b^{-1})\,\theta_{n_b}, \tag{1.4}$$

where

$$\begin{aligned} \theta_m &= 2^{1-m} && \text{if } 1 \leqslant m \leqslant 12, \\ &= (2m^2(2\log m + \log\log m + 3))^{-1} && \text{if } m > 12. \end{aligned}$$

Then $\{n_i\}_{i=1}^{s}$ *has property* (P^*) *and hence* (P).

Theorem 2 is easily demonstrated by elementary argument (Section 2). The proof of Theorem 3 is a generalization of Vaughan's method [8] to mixed powers. His Lemma 11 is modified to the form of our Lemma 10. In order to simplify the proof, the type of density argument of Vaughan [8, Sect. 6] is replaced by a simpler but weaker one of Davenport and Roth [3, Lemma 3].

2. Proof of Theorem 2

For the sake of clarity, we write $v_i = n_i^{-1}$, $i = 1, 2, \ldots$. Suppose that $n_1, n_2, \ldots, n_s$ are positive integers such that $\sum_{i=1}^{s} v_i < 1$. Let

$$\chi = 1 - \sum_{i=1}^{s} v_i$$

and

$$0 < \varepsilon < \left\{2\left(1 + \frac{1}{n_1\chi}\right)\right\}^{-1}.$$

For each positive integer x_1, put

$$M_{x_1} = \bigcup I(x_1, x_2, \ldots, x_s) \qquad (x_2, \ldots, x_s > 0,\ I \cap [0, 1] \neq \varnothing),$$

where $I(x_1, x_2, \ldots, x_s)$ denotes the open interval with end points at $(x_2^{n_2} + \cdots + x_s^{n_s} \pm \varepsilon)/x_1^{n_1}$ and $x_2, \ldots, x_s$ are positive integers. For each x_1, if $I(x_1, x_2, \ldots, x_s) \cap [0, 1] \neq \varnothing$, then

$$x_i \leqslant x_1^{n_1 v_i}, \qquad i = 2, \ldots, s.$$

By the crudest estimate, we have,

$$\text{measure of} \qquad M_{x_1} \leqslant \frac{2\varepsilon}{x_1^{n_1}}\, x_1^{-n_1\chi + n_1 - 1} = \frac{2\varepsilon}{x_1^{n_1\chi + 1}}.$$

Consequently,

$$\text{measure of} \quad \bigcup_{x_1=1}^{\infty} M_{x_1} \leqslant \sum_{x_1=1}^{\infty} \frac{2\varepsilon}{x_1^{n_1\chi+1}} < 1.$$

This guarantees the existence of an irrational number $\lambda \in [0, 1]\backslash \bigcup_{x_1=1}^{\infty} M_{x_1}$ for which we have

$$|\lambda x_1^{n_1} - x_2^{n_2} - \cdots - x_s^{n_s}| \geqslant \varepsilon$$

for any $x_1, x_2, ..., x_s > 0$. Hence (1.1) is not solvable with $\lambda_1 = \lambda$, $\lambda_2 = \cdots = \lambda_s = -1$ and $\eta = 0$. This proves Theorem 2.

We come now to the proof of Theorem 3.

3. Notation and Definitions

Let $\{n_i\}_{i=1}^{s}$ $(s \geqslant 2)$ be a sequence of positive integers satisfying the hypotheses of Theorem 3 and η is a real number. Given $\{\lambda_i\}_{i=1}^{s}$, we may assume (cf. [1, p. 143, Sect. 2]) that λ_h/λ_l is both irrational and negative for some $1 \leqslant h < l \leqslant s$. Throughout, α and z are real variables while p and x, y, with or without suffices, are primes and integers, respectively. δ is a sufficiently small positive number whose choice depends on almost everything except X and c is a positive constant not necessarily the same in each occurrence. Recall the notation $v_i = n_i^{-1}$ and for simplicity, write $e(\alpha)$ for $\exp(i2\pi\alpha)$.

Let

$$T_1 = \{1, 2, ..., a\}\backslash\{h, l\}, \tag{3.1}$$

$$T_2 = \{a+1, a+2, ..., b\}\backslash\{h, l\},$$

$$U = \{b+1, b+2, ..., c\}\backslash\{h, l\} = \{u_i\colon 1 \leqslant i \leqslant d \text{ and } u_i < u_j \text{ if } i < j\},$$

$$W = \{c+1, c+2, ..., s\}\backslash\{h, l\} = \{w_i\colon 1 \leqslant i \leqslant e \text{ and } w_i < w_j \text{ if } i < j\},$$

$$\kappa_i = \sum_{t \in T_i} v_t, \qquad i = 1, 2, \tag{3.2}$$

$$\mu_u = \prod_{i=1}^{d} (1 - v_{u_i}), \qquad \mu_w = \prod_{i=1}^{e} (1 - v_{w_i}), \tag{3.3}$$

and

$$v = v_h + v_l + \kappa_1 + \kappa_2 - \mu_u - \mu_w + 1. \tag{3.4}$$

By (1.2), (1.4) and (3.3), we have

$$\mu_u < (n_a + 4)\, v_a \theta_{n_a} \qquad \text{and} \qquad \mu_w < (n_b + 4)\, v_b \theta_{n_b}. \tag{3.5}$$

Since λ_h/λ_l is irrational, by Theorem 193 in [5], there exist infinitely many pairs of integers a, q such that $q \geqslant 1$, $(a, q) = 1$ and $|\lambda_h/\lambda_l - a/q| < 1/2q^2$. Let

$$X = q^{1/(1-\sigma_2-\sigma_3)}, \tag{3.6}$$

where σ_2, σ_3 are positive numbers satisfying

$$\sigma_2 < 2v_l/9, \qquad \sigma_3 < v_h/3 \qquad \text{and} \qquad \sigma_2 + \sigma_3 < \tfrac{1}{2}. \tag{3.7}$$

Let

$$\pi_m = (2^{2m+2}(m+1))^{-1}, \qquad m \geqslant 1, \tag{3.8}$$

$$\tau = X^{-\sigma_1}, \tag{3.9}$$

where

$$0 < \sigma_1 < \min(\pi_{n_l}\sigma_2, \pi_{n_h}\sigma_3, \tfrac{1}{2} - (\sigma_2 + \sigma_3)), \tag{3.10}$$

and

$$\begin{aligned} K_\tau(\alpha) &= \tau^2 && \text{if} \quad \alpha = 0, \\ &= ((\sin \tau\pi\alpha)/\pi\alpha)^2 && \text{if} \quad \alpha \neq 0. \end{aligned} \tag{3.11}$$

For $t \in T_1 \cup T_2 \cup \{h, l\}$, define

$$f_t(\alpha) = \sum e(\alpha\lambda_t p^{n_t}) \qquad (\rho_t X^{v_t} < p \leqslant 2\rho_t X^{v_t}), \tag{3.12}$$

and

$$g_t(\alpha) = \sum e(\alpha\lambda_t x^{n_t}) \qquad (\rho_t X^{v_t} < x \leqslant 2\rho_t X^{v_t}), \tag{3.13}$$

where

$$\left.\begin{aligned} &\rho_h = 1, \qquad \rho_l = \frac{1}{2}\left(\left|\frac{\lambda_h}{\lambda_l}\right| \left(2^{n_h} - \frac{1}{3}\right)\right)^{v_l} \\ &\text{and} \\ &\rho_t = \frac{1}{2}\left(\left|\frac{\lambda_h}{\lambda_t}\right| \frac{1}{4b}\right)^{v_t} \qquad \text{if} \quad t \neq h, l. \end{aligned}\right\} \tag{3.14}$$

For any positive Z and any positive integer k, write

$$S(\alpha, Z, k) = \sum e(\alpha p^k) \qquad (Z < p \leqslant 2Z) \tag{3.15}$$

and define

$$\Psi_u(\alpha) = S(\lambda_{u_1}\alpha, \delta X^{v_{u_1}}, n_{u_1}) \prod_{i=2}^{d} S(\lambda_{u_i}\alpha, X^{\Delta_i}, n_{u_i}), \tag{3.16}$$

$$\Psi_w(\alpha) = S(\lambda_{w_1}\alpha, \delta X^{v_{w_1}}, n_{w_1}) \prod_{i=2}^{e} S(\lambda_{w_i}\alpha, X^{\Lambda_i}, n_{w_i}), \tag{3.17}$$

where

$$\Delta_i = v_{u_i} \prod_{m=1}^{i-1} (1 - v_{u_m}), \qquad i = 2,..., d, \tag{3.18}$$

and

$$\Lambda_i = v_{w_i} \prod_{m=1}^{i-1} (1 - v_{w_m}), \qquad i = 2,..., e. \tag{3.19}$$

Let $F_i(\alpha) = \prod_{t \in T_i} f_t(\alpha)$ $(i = 1, 2)$, $F(\alpha) = F_1(\alpha) F_2(\alpha)$ and $\Psi(\alpha) = \Psi_u(\alpha)\, \Psi_w(\alpha)$. We dissect the real line into the following three regions:

$$\left.\begin{aligned} E_1 &= \{\alpha : |\alpha| \leqslant |\lambda_l|^{-1} X^{\sigma_2 - 1}\}, \\ E_2 &= \{\alpha : |\lambda_l|^{-1} X^{\sigma_2 - 1} < |\alpha| \leqslant X^{1 - 2(\sigma_2 + \sigma_3)}\}, \\ E_3 &= \{\alpha : X^{1 - 2(\sigma_2 + \sigma_3)} < |\alpha|\}. \end{aligned}\right\} \tag{3.20}$$

For sinplicity, we write $W_i = \int_{E_i} f_h(\alpha) f_l(\alpha)\, F(\alpha)\, \Psi(\alpha)\, e(\alpha\eta)\, K_\tau(\alpha)\, d\alpha$ $(i = 1, 2, 3)$, $W = W_1 + W_2 + W_3$ and $L = \log X$. The constants implied in the Vinogradov's symbols $\ll$ and $\gg$ are independent of X. In the proof, we always assume that X is sufficiently large (whose choice depends on δ) and we shall show that $W_1 \gg \tau^2 X^v L^{-c}$, $W_2, W_3 \ll \tau^2 X^{v-\delta}$ and hence $W \gg \tau^2 X^v L^{-c}$.

4. Preliminary Lemmas

Lemma 1. *For any real z,*

$$\int_{-\infty}^{\infty} e(\alpha z)\, K_\tau(\alpha)\, d\alpha = \max(0, \tau - |z|).$$

Proof. By (3.11) this is Lemma 4 of Davenport and Heilbronn [2].

LEMMA 2. *For any $t \in T_1 \cup T_2 \cup \{h, l\}$,*

$$f_t(\alpha) \ll X^{v_t}. \tag{4.1}$$

Furthermore,

$$\Psi_u(\alpha) \ll X^{1-\mu_u} \qquad \text{and} \qquad \Psi_w(\alpha) \ll X^{1-\mu_w}. \tag{4.2}$$

Proof. Inequality (4.1) follows from (3.12). From (3.18) and $(3.3)_1$,

$$v_{u_1} + \sum_{i=2}^{d} \Delta_i = 1 - \mu_u. \tag{4.3}$$

Hence, by (3.15) and (3.16), $\Psi_u(\alpha) \ll X^{v_{u_1}} \prod_{i=2}^{d} X^{\Delta_i} = X^{1-\mu_u}$. The proof of $(4.2)_2$ is similar.

LEMMA 3. *For $m = u$ or w,*

$$\int_{-\infty}^{\infty} |\Psi_m(\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{1-\mu_m}.$$

Proof. By (3.15), (3.16), (3.17) and (4.3), this can be shown in the same way as Lemma 3 of Davenport and Roth [3].

LEMMA 4. *Let $m = u$ or w and $t \in T_1 \cup T_2$. If $r \geqslant 4 + n_t$ and $r > n_t \mu_m / \theta_{n_t}$, then*

$$\int_{-\infty}^{\infty} |g_t^r(\alpha)\, \Psi_m^2(\alpha)|\, K_\tau(\alpha)\, d\alpha \ll \tau X^{r v_t - 2\mu_m + 1}.$$

Proof. When $n_t \geqslant 2$, by Lemma 3 and (4.2), this can be shown in the same way as Theorem 1 of Vaughan [8]. The arguments still work when $n_t = 2$ or 3. If $n_t = 1$, by Lemma 1,

$$\begin{aligned}\int_{-\infty}^{\infty} |g_t(\alpha)|^2 K_\tau(\alpha)\, d\alpha &= \sum \max(0, \tau - |\lambda_t(x-y)|) \qquad (\rho_t X < x, y \leqslant 2\rho_t X)\\ &\ll \tau X.\end{aligned}$$

Therefore, by (3.13) and (4.2),

$$\begin{aligned}\int_{-\infty}^{\infty} |g_t^r(\alpha)\, \Psi_m^2(\alpha)|\, K_\tau(\alpha)\, d\alpha &\ll X^{r-2} X^{2(1-\mu_m)} \int_{-\infty}^{\infty} |g_t(\alpha)|^2 K_\tau(\alpha)\, d\alpha\\ &\ll \tau X^{r-2\mu_m+1}.\end{aligned}$$

This completes the proof of Lemma 4.

LEMMA 5.

$$\int_{-\infty}^{\infty} |F_1(\alpha)\, \Psi_u(\alpha)|^2\, K_\tau(\alpha)\, d\alpha \ll \tau X^{2\kappa_1 - 2\mu_u + 1} \tag{4.4}$$

and

$$\int_{-\infty}^{\infty} |F_2(\alpha)\, \Psi_w(\alpha)|^2\, K_\tau(\alpha)\, d\alpha \ll \tau X^{2\kappa_2 - 2\mu_w + 1}. \tag{4.5}$$

Proof. By Lemma 1, it is clear that

$$\int_{-\infty}^{\infty} |F_1(\alpha)\, \Psi_u(\alpha)|^2\, K_\tau(\alpha)\, d\alpha \leqslant \int_{-\infty}^{\infty} \left| \left(\prod_{t \in T_1} g_t(\alpha) \right) \Psi_u(\alpha) \right|^2 K_\tau(\alpha)\, d\alpha.$$

From $(1.3)_1$, (1.2) and (3.1), we have $\sum_{t \in T_1} 2(4 + n_t)^{-1} \geqslant 1$ and hence there is a non-negative number Q so that $\sum_{t \in T_1} 2(Q + 4 + n_t)^{-1} = 1$. Since θ_m decreases with m, by $(3.5)_1$, (3.1) and (1.2), $(Q + 4 + n_t) > n_t \mu_u / \theta_{n_t}$ when $t \in T_1$. It follows from Lemma 4 that

$$\int_{-\infty}^{\infty} |(g_t(\alpha))^{(Q+4+n_t)}\, \Psi_u^2(\alpha)|\, K_\tau(\alpha)\, d\alpha \ll \tau X^{(Q+4+n_t)v_t - 2\mu_u + 1}.$$

Hence, by Hölder's inequality and (3.2), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \left| \left(\prod_{t \in T_1} g_t(\alpha) \right) \Psi_u(\alpha) \right|^2 K_\tau(\alpha)\, d\alpha \\ &\leqslant \prod_{t \in T_1} \left\{ \int_{-\infty}^{\infty} |(g_t(\alpha))^{(Q+4+n_t)}\, \Psi_u^2(\alpha)|\, K_\tau(\alpha)\, d\alpha \right\}^{2/(Q+4+n_t)} \\ &\ll \prod_{t \in T_1} (\tau X^{(Q+4+n_t)v_t - 2\mu_u + 1})^{2/(Q+4+n_t)} \\ &= \tau X^{2\kappa_1 - 2\mu_u + 1}. \end{aligned}$$

This completes the proof of (4.4). The proof of (4.5) is similar.

5. THE REGION E_1

Let $\rho = \beta + i\gamma$ be a non-trivial zero of the Riemann zeta function. For $b = h$ or l, let $\Phi_b = X^{v_b/3}$,

$$E_\rho(\alpha) = \sum \frac{e(\alpha \lambda_b x)\, x^{\rho v_b - 1}}{\log x} \qquad (\rho_b^{n_b} X < x \leqslant (2\rho_b)^{n_b} X)$$

and $J_b(\alpha) = \Sigma' E_\rho(\alpha)$, where Σ' denotes the summation over all ρ with $|\gamma| \leqslant \Phi_b$ and $\beta \geqslant 2/3$. Let

$$I_b(\alpha) = \int_{\rho_b^{n_b} X}^{(2\rho_b)^{n_b} X} \frac{e(\alpha \lambda_b z)\, z^{v_b - 1}}{\log z} dz \tag{5.1}$$

and

$$B_b(\alpha) = f_b(\alpha) - I_b(\alpha) + J_b(\alpha). \tag{5.2}$$

LEMMA 6.

$$I_b(\alpha) \ll X^{v_b} \min(1, |\alpha|^{-1} X^{-1}), \tag{5.3}$$

$$\int_{-1/2}^{1/2} |J_b(\alpha)|^2 \, d\alpha \ll X^{2v_b - 1} \exp(-2L^{1/5}),$$

$$\int_{-1/2}^{1/2} |I_b(\alpha)|^2 \, d\alpha \ll X^{2v_b - 1},$$

$$\int_{E_1} |B_b(\alpha)|^2 \, d\alpha \ll X^{2v_b - 1} \exp(-2L^{1/5})$$

and

$$\int_{E_1} |f_b(\alpha)|^2 \, d\alpha \ll X^{2v_b - 1}.$$

Proof. This can be proved in the same way as Lemma 8 of Vaughan [8].

LEMMA 7.

$$\int_{E_1} f_h(\alpha) f_l(\alpha)\, F(\alpha)\, \Psi(\alpha)\, e(\alpha\eta)\, K_\tau(\alpha)\, d\alpha$$
$$- \int_{-\infty}^{\infty} I_h(\alpha)\, I_l(\alpha)\, F(\alpha)\, \Psi(\alpha)\, e(\alpha\eta)\, K_\tau(\alpha)\, d\alpha$$
$$\ll \tau^2 X^v \exp(-L^{1/5}).$$

Proof. By (5.2), Hölder's inequality and Lemma 6, we have

$$\int_{E_1} |f_h(\alpha) f_l(\alpha) - I_h(\alpha)\, I_l(\alpha)| \, d\alpha$$
$$\ll \int_{E_1} (|f_h(\alpha)\, B_l(\alpha)| + |f_h(\alpha)\, J_l(\alpha)| + |I_l(\alpha)\, B_h(\alpha)| + |I_l(\alpha)\, J_h(\alpha)|) \, d\alpha$$
$$\ll X^{v_h + v_l - 1} \exp(-L^{1/5}).$$

Hence, in view of (3.11) and (3.4),

$$\int_{E_1} f_h(\alpha) f_l(\alpha)\, F(\alpha)\, \Psi(\alpha)\, e(\alpha\eta)\, K_\tau(\alpha)\, d\alpha$$

$$- \int_{E_1} I_h(\alpha)\, I_l(\alpha)\, F(\alpha)\, \Psi(\alpha)\, e(\alpha\eta)\, K_\tau(\alpha)\, d\alpha$$

$$\ll \tau^2 X^{\kappa_1+\kappa_2-\mu_u-\mu_w+2} \int_{E_1} |f_h(\alpha) f_l(\alpha) - I_h(\alpha)\, I_l(\alpha)|\, d\alpha$$

$$\ll \tau^2 X^v \exp(-L^{1/5}). \tag{5.4}$$

Furthermore, from (5.3) and $(3.20)_1$,

$$\int_{R\setminus E_1} I_h(\alpha)\, I_l(\alpha)\, F(\alpha)\, \Psi(\alpha)\, e(\alpha\eta)\, K_\tau(\alpha)\, d\alpha$$

$$\ll \tau^2 X^{\kappa_1+\kappa_2-\mu_u-\mu_w+2} \int_{R\setminus E_1} |I_h(\alpha)\, I_l(\alpha)|\, d\alpha$$

$$\ll \tau^2 X^{v-1} \int_{R\setminus E_1} \frac{d\alpha}{\alpha^2} \ll \tau^2 X^{v-\sigma_2}.$$

This together with (5.4) proves Lemma 7.

LEMMA 8.

$$W_1 \gg \tau^2 X^v L^{-c}.$$

Proof. In view of Lemma 7, it suffices to show that $\int_{-\infty}^{\infty} I_h(\alpha)\, I_l(\alpha)\, F(\alpha)\, \Psi(\alpha)\, e(\alpha\eta)\, K_\tau(\alpha)\, d\alpha \gg \tau^2 X^v L^{-c}$. By definitions of $F(\alpha)$ and $\Psi(\alpha)$, we may write $F(\alpha)\, \Psi(\alpha)\, e(\alpha\eta) = \sum_\phi e(\alpha\phi)$, where

$$\phi = \eta + \sum_{t \in T_1 \cup T_2} \lambda_t p_t^{n_t} + \sum_{1 \leqslant i \leqslant d} \lambda_{u_i} p_{u_i}^{n_{u_i}} + \sum_{1 \leqslant i \leqslant e} \lambda_{w_i} p_{w_i}^{n_{w_i}},$$

$$\left.\begin{array}{lll} & \rho_t X^{v_t} < p_t \leqslant 2\rho_t X^{v_t}, & t \in T_1 \cup T_2, \\ \delta X^{v_{u_1}} < p_{u_1} \leqslant 2\delta X^{v_{u_1}}, & X^{\Delta_i} < p_{u_i} \leqslant 2X^{\Delta_i}, & i = 2,\dots, d, \\ \delta X^{v_{w_1}} < p_{w_1} \leqslant 2\delta X^{v_{w_1}}, & X^{\Lambda_i} < p_{w_i} \leqslant 2X^{\Lambda_i}, & i = 2,\dots, e, \end{array}\right\} \tag{5.5}$$

and Δ_i, Λ_i are defined in (3.18) and (3.19). Clearly, when X is sufficiently large,

$$\left| \sum_{1 \leqslant i \leqslant d} \lambda_{u_i} p_{u_i}^{n_{u_i}} + \sum_{1 \leqslant i \leqslant e} \lambda_{w_i} p_{w_i}^{n_{w_i}} \right| < c\delta X.$$

Hence, by $(3.14)_3$ and assuming that $|\eta| < \delta X$, we have

$$|\phi| < \tfrac{5}{16}|\lambda_h| X. \tag{5.6}$$

On the other hand, from the prime number theorem, (4.3) and (3.2), we have

$$\sum_{\phi} 1 \gg X^{\kappa_1+\kappa_2-\mu_u-\mu_w+2} \cdot L^{-c}. \tag{5.7}$$

If $\mu = \{3(2^{n_l-1})/(2^{n_h}-\frac{1}{3})\}^{v_l}$, then $1 < \mu < 2$ and hence

$$[(\mu\rho_l)^{n_l} X, (2\rho_l)^{n_l} X] \subset [\rho_l^{n_l} X, (2\rho_l)^{n_l} X].$$

Therefore, with the help of Lemma 1 and (5.1),

$$\begin{aligned}\int_{-\infty}^{\infty} & I_h(\alpha)\, I_l(\alpha)\, F(\alpha)\, \Psi(\alpha)\, e(\alpha\eta)\, K_\tau(\alpha)\, d\alpha \\ & \gg \sum_{\phi} \int_X^{2^{n_h}X} \int_{(\mu\rho_l)^{n_l}X}^{(2\rho_l)^{n_l}X} \left\{\frac{z_h^{v_h-1} z_l^{v_l-1}}{(\log z_h)(\log z_l)}\right\} \\ & \quad \times \max(0, \tau - |\lambda_h z_h + \lambda_l z_l + \phi|)\, dz_l\, dz_h. \end{aligned} \tag{5.8}$$

Let $\phi^* = \lambda_h z_h + \lambda_l z_l + \phi$. For any $z_l \in [(\mu\rho_l)^{n_l} X, (2\rho_l)^{n_l} X]$, if z_h varies over an interval of length $\tau|\lambda_h|^{-1}$ so that $|\phi^*| < \tau/2$, then, by (5.6) and the fact that $\lambda_h/\lambda_l < 0$,

$$\begin{aligned} z_h &= \left|\frac{\lambda_l}{\lambda_h}\right| z_l - \frac{\phi}{\lambda_h} + \frac{\phi^*}{\lambda_h} \\ &\leqslant \left|\frac{\lambda_l}{\lambda_h}\right| (2\rho_l)^{n_l} X + \frac{1}{|\lambda_h|}\left(\frac{5}{16}|\lambda_h| X\right) + \frac{1}{|\lambda_h|} < 2^{n_h} X \end{aligned}$$

and

$$z_h \geqslant \left|\frac{\lambda_l}{\lambda_h}\right| (\mu\rho_l)^{n_l} X - \frac{1}{|\lambda_h|}\left(\frac{5}{16}|\lambda_h| X\right) - \frac{1}{|\lambda_h|} > X.$$

This and (5.7) show that the right-hand side of (5.8) is

$$\gg \frac{X^{v_h+v_l-2}}{(\log X)^2} X\tau|\lambda_h|^{-1} \frac{\tau}{2} \sum_{\phi} 1 \gg \tau^2 X^v L^{-c}$$

and the proof of Lemma 8 is completed.

6. The Region E_2

Lemma 9. *Let k be a positive integer. For any real θ, suppose that*

$$\left|\theta - \frac{x}{y}\right| \leqslant \frac{1}{y^2}, \qquad (x, y) = 1, \qquad Y = \min(Z^{1/3}, y, Z^k y^{-1})$$

and $\log Y \geqslant 2^{6k-2}(2k+1) \log\log Z$. *Then*

$$\sum_{p \leqslant Z} e(\theta p^k) \ll Z Y^{-\pi_k},$$

where π_k is defined in (3.8).

Proof. This is a theorem of Vinogradov [9].

Lemma 10. *For any $\alpha \in E_2$, either $f_h(\alpha) \ll X^{v_h - \pi_{n_h}\sigma_3 + \delta}$ or $f_l(\alpha) \ll X^{v_l - \pi_{n_l}\sigma_2 + \delta}$.*

Proof. Let $\alpha \in E_2$. By Theorem 36 in [5], we choose integers a_h, q_h, a_l, q_l such that

$$(a_h, q_h) = 1, \qquad 1 \leqslant q_h \leqslant X^{1-\sigma_3}, \qquad \lambda_h \alpha = \frac{a_h}{q_h} + \beta_h,$$

$$|\beta_h| \leqslant \frac{1}{q_h X^{1-\sigma_3}} \leqslant \frac{1}{q_h^2},$$

and

$$(a_l, q_l) = 1, \qquad 1 \leqslant q_l \leqslant X^{1-\sigma_2}, \qquad \lambda_l \alpha = \frac{a_l}{q_l} + \beta_l,$$

$$|\beta_l| \leqslant \frac{1}{q_l X^{1-\sigma_2}} \leqslant \frac{1}{q_l^2}.$$

If $q_h \geqslant X^{\sigma_3 - \delta}$, then in view of $(3.7)_2$, $\min(X^{v_h/3}, q_h, X q_h^{-1}) \geqslant X^{\sigma_3 - \delta}$. Hence, by Lemma 9 with $Z = X^{v_h}$, $x = a_h$, $y = q_h$, $\theta = \lambda_h \alpha$ and $k = n_h$, we have $f_h(\alpha) \ll X^{v_h - \pi_{n_h}\sigma_3 + \delta}$.

Similarly, if $q_l \geqslant X^{\sigma_2 - \delta}$, then $f_l(\alpha) \ll X^{v_l - \pi_{n_l}\sigma_2 + \delta}$. By the method of Davenport and Heilbronn [2, Lemma 13], we can show that $q_h < X^{\sigma_3 - \delta}$ and $q_l < X^{\sigma_2 - \delta}$ contradicts (3.6). This completes the proof of the lemma.

Lemma 11. $W_2 \ll \tau^2 X^{v - \delta}$.

Proof. Let $b=h$ or l. By (3.12), Hölder's inequality and Lemma 5,

$$\int_{-\infty}^{\infty} |f_b(\alpha)\,F(\alpha)\,\Psi(\alpha)|\,K_\tau(\alpha)\,d\alpha$$
$$\ll X^{v_b}\left\{\int_{-\infty}^{\infty} |F_1(\alpha)\,\Psi_u(\alpha)|^2\,K_\tau(\alpha)\,d\alpha\right\}^{1/2}\left\{\int_{-\infty}^{\infty} |F_2(\alpha)\,\Psi_w(\alpha)|^2\,K_\tau(\alpha)\,d\alpha\right\}^{1/2}$$
$$\ll \tau X^{v_b+\kappa_1+\kappa_2-\mu_u-\mu_w+1}.$$

Hence from Lemma 10, (3.9) and (3.10),

$$\begin{aligned}|W_2| &\leqslant \int_{E_2} |f_h(\alpha) f_l(\alpha)|\,|F(\alpha)\,\Psi(\alpha)|\,K_\tau(\alpha)\,d\alpha\\ &\ll X^{v_h-\pi_{n_h}\sigma_3+\delta}\int_{-\infty}^{\infty} |f_l(\alpha)\,F(\alpha)\,\Psi(\alpha)|\,K_\tau(\alpha)\,d\alpha\\ &\quad + X^{v_l-\pi_{n_l}\sigma_2+\delta}\int_{-\infty}^{\infty} |f_h(\alpha)\,F(\alpha)\,\Psi(\alpha)|\,K_\tau(\alpha)\,d\alpha\\ &\ll \tau^2 X^{v-\delta}.\end{aligned}$$

This proves Lemma 11.

7. The Region E_3

Lemma 12. *Let $\Omega(\alpha)=\Sigma e(\alpha\omega(z_1, z_2,..., z_m))$, where $\omega(z_1, z_2,..., z_m)$ is any real valued function and the summation is over any finite set of values of $z_1, z_2,..., z_m$. Then for any $Z>4\tau^{-1}$ we have*

$$\int_{|\alpha|>Z} |\Omega(\alpha)|^2\,K_\tau(\alpha)\,d\alpha \leqslant \frac{16}{Z\tau}\int_{-\infty}^{\infty} |\Omega(\alpha)|^2\,K_\tau(\alpha)\,d\alpha.$$

Proof. This is Lemma 13 of Vaughan [8].

Lemma 13. $W_3 \ll \tau^2 X^{v-\delta}$.

Proof. By (3.12), Hölder's inequality, $(3.20)_3$, Lemmas 12 and 5, (3.9) and (3.10),

$$
\begin{aligned}
|W_3| &\leqslant \int_{E_3} |f_h(\alpha) f_l(\alpha) F(\alpha)\, \Psi(\alpha)|\, K_\tau(\alpha)\, d\alpha \\
&\ll X^{v_h+v_l} \left\{ \int_{E_3} |F_1(\alpha)\, \Psi_u(\alpha)|^2\, K_\tau(\alpha)\, d\alpha \right\}^{1/2} \\
&\quad \times \left\{ \int_{E_3} |F_2(\alpha)\, \Psi_w(\alpha)|^2\, K_\tau(\alpha)\, d\alpha \right\}^{1/2} \\
&\ll X^{v_h+v_l+\sigma_1+2(\sigma_2+\sigma_3)-1} \left\{ \int_{-\infty}^{\infty} |F_1(\alpha)\, \Psi_u(\alpha)|^2\, K_\tau(\alpha)\, d\alpha \right\}^{1/2} \\
&\quad \times \left\{ \int_{-\infty}^{\infty} |F_2(\alpha)\, \Psi_w(\alpha)|^2\, K_\tau(\alpha)\, d\alpha \right\}^{1/2} \\
&\ll \tau^2 X^{v-\delta}.
\end{aligned}
$$

8. Completion of the Proof of Theorem 3

As a consequence of Lemmas 8, 11, and 13, we have $W \gg \tau^2 X^v L^{-c}$. On the other hand, by Lemma 1,

$$
W = \sum \max\left(0, \tau - \left|\eta + \sum_{i=1}^{s} \lambda_i p_i^{n_i}\right|\right),
$$

where the summation is over the s-tuples $P = (p_1, p_2, \ldots, p_s)$ satisfying

$$
\rho_b X^{v_b} < p_b \leqslant 2\rho_b X^{v_b}, \qquad b = h, l \qquad \text{and} \qquad (5.5). \tag{8.1}
$$

Therefore $W \leqslant \tau N$, where N is the number of solutions of the inequality

$$
\left|\eta + \sum_{i=1}^{s} \lambda_i p_i^{n_i}\right| < \tau
$$

with P satisfying (8.1). Since

$$
\tau = X^{-\sigma_1} \ll (\max p_i)^{-n_1\sigma_1},
$$

the number of solutions of the inequality

$$
\left|\eta + \sum_{i=1}^{s} \lambda_i p_i^{n_i}\right| < (\max p_i)^{-\sigma},
$$

where $0 < \sigma < n_1\sigma_1$, is $\gg N \gg \tau X^v L^{-c}$ which tends to infinity with X. This completes the proof of our Theorem 3.

9. Proof of Theorem 1

If $\{n_i\}_{i=1}^{\infty}$ is a sequence of positive integers such that $\sum_{i=1}^{\infty} n_i^{-1} = \infty$, then for any given positive integer j, $\sum_{i=j}^{\infty} n_i^{-1} = \infty$. This ensures the existence of an integer s so that the sequence $\{n_i\}_{i=j}^{j+s-1}$, after the necessary renumbering of its terms for fulfilment of condition (i), satisfies condition (ii) and hence the hypotheses of Theorem 3. From this the sufficiency of the condition $\sum_{i=1}^{\infty} n_i^{-1} = \infty$ follows immediately.

Conversely, if $\sum_{i=1}^{\infty} n_i^{-1} < \infty$, choose j large enough so that $\sum_{i=j}^{\infty} n_i^{-1} < 1$, then the necessity of the condition $\sum_{i=1}^{\infty} n_i^{-1} = \infty$ follows from Theorem 2. This completes the proof of Theorem 1.

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